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Cosmological perturbations from inflation

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Abstract
Matter is distributed very homogeneously and isotropically on scales larger than a few hundred Mpc. The measurements of the microwave background temperature fluctuations show that at recombination the universe was extremely homogeneous and isotropic (with accuracy $\sim 10^{-4}$) on all scales up to the present horizon (Spergel *et al* 2006 *Preprint* astro-ph/0603449; MacTavish *et al* 2005 *Preprint* astro-ph/0507503). On the other hand, there is a large scale structure in the observable universe and one of the central issues of contemporary cosmology is the explanation of the origin of primordial inhomogeneities, which serve as the seeds for structure formation. Before the advent of inflationary cosmology the initial perturbations were *postulated* and their spectrum was designed to fit observational data. In this way practically any observation could be 'explained', or more accurately *described, by* arranging the appropriate initial conditions. In contrast, inflationary cosmology *truly explains* the origin of primordial inhomogeneities and *predicts* their spectrum (Mukhanov and Chibisov 1981 *JETP Lett.* **33** 532; Mukhanov and Chibisov 1982 *Sov. Phys.—JETP* **56** 258). Thus it becomes possible to test this theory by comparing its predictions with observations. According to cosmic inflation, primordial perturbations originated from quantum fluctuations. These fluctuations have substantial amplitudes only on scales close to the Planckian length, but during the inflationary stage they are stretched to galactic scales with nearly unchanged amplitudes. Thus, inflation links the large-scale structure of the universe to its microphysics. The resulting spectrum of inhomogeneities is not very sensitive to the details of any particular inflationary scenario and has nearly universal shape. This leads to concrete predictions for the spectrum of cosmic microwave background anisotropies.

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1. Inflation

Inflation is the stage of accelerated expansion with graceful exit to the decelerating Friedmann Universe. To describe the stage of cosmic inflation with subsequent graceful exit one usually considers the slow-rolling scalar field which imitates the equation of state $p \approx -\varepsilon$. Instead of considering the concrete scalar field models we can use, however, the language of ideal hydrodynamics, which is an adequate phenomenological description of matter on large scales both before and after inflation. To parametrize the 'hydrodynamical fluid' it is natural to use the scalar variable φ and write the most generic action, which leads to the second-order equation for a scalar variable, in the form

$$S = \int p(X,\varphi)\sqrt{-g}\,\mathrm{d}x, \tag{1.1}$$

where p is an arbitrary function of φ and $X \equiv \frac{1}{2}(\partial_\mu\varphi\partial\varphi^\mu)$. Variation of (1.1) with respect to the metric gives the energy–momentum tensor, which, for $X > 0$, can be written in the form of 'ideal hydrodynamical fluid':

$$T^\mu_\nu = (\varepsilon + p)u^\mu u_\nu - p\delta^\mu_\nu, \tag{1.2}$$

where the Lagrangian p plays the role of the effective pressure and

$$\varepsilon = 2X\frac{\partial p}{\partial X} - p, \qquad u_\nu = \frac{\partial_\nu\varphi}{\sqrt{2X}}. \tag{1.3}$$

Lagrangian (1.1) can be used to describe the usual hydrodynamical fluid as well as the matter with the equation of state imitating the cosmological constant. For example for $p = X^2$ we have ultrarelativistic equation of state $p = \varepsilon/3$. On the other hand if p satisfies the condition $X\partial p/\partial X \ll p$ for some range of X and φ, the equation of state is $p \approx -\varepsilon$ and we have an inflationary solution. All successful simple inflationary scenarios with the scalar field, e.g. [7, 13], can be described by the action of the form (1.1). Moreover, using conformal equivalence of the higher derivative gravity to Einstein gravity with an extra scalar field [10] we can describe inflation in the higher derivative gravity [11, 12] in the same way as in the theories with the scalar field.

One could consider the Lagrangians with two or more scalar fields. Then the number of options increases. The most popular model of this type, which has some justification in particle physics is the so-called hybrid inflation [14]. However, in the case of several scalar fields the inflation drastically loses its predictive power and therefore I will not consider this situation here.

Action (1.1) can be used to derive the spectrum of inflationary perturbations in generic inflationary models entirely in terms of hydrodynamical quantities p and ε, characterizing the state of the matter during inflation.

There are two ways to realize the condition which leads to inflation, $X\partial p/\partial X \ll p$, namely, either keeping the kinetic term X small (slow-roll inflation) [7] or taking the nontrivial dependence of the Lagrangian on X, so that the derivative $X\partial p/\partial X \ll p$ even for large X (k-inflation) [8]. Below we will derive the spectrum of the inflationary perturbations in the general case for an arbitrary equation of state.

2. Scalar perturbations

2.1. Equations

Let us consider a flat universe filled by matter described by action (1.1). If p depends only on X, then $\varepsilon = \varepsilon(X)$, and in many cases equation (1.3) can be rearranged to give $p = p(\varepsilon)$, the equation of state for an isentropic fluid. In the general case, $p = p(X,\varphi)$, the pressure cannot be expressed only in terms of ε since X and φ are independent. However, even in this case, the hydrodynamical analogy is still useful. For a canonical scalar field we have $p = X - V(\varphi)$ and, correspondingly, $\varepsilon = X + V$.

To derive the equations for scalar perturbations we will follow [15] and work in the conformal-Newtonian coordinate system, where the metric takes the form

$$ds^2 = a^2(\eta)[(1+2\Phi)\,d\eta^2 - (1-2\Psi)\delta_{ik}\,dx^i\,dx^k]. \tag{2.1}$$

The state of a flat, homogeneous universe is characterized completely by the scale factor $a(\eta)$ and the homogeneous field $\varphi_0(\eta)$, which satisfy the familiar equations

$$\mathcal{H}^2 = \frac{8\pi}{3}a^2\varepsilon, \tag{2.2}$$

$$\varepsilon' = \varepsilon_{,X}X_0' + \varepsilon_{,\varphi}\varphi_0' = -3\mathcal{H}(\varepsilon + p), \tag{2.3}$$

where $\mathcal{H} \equiv a'/a$, the prime denotes the derivative with respect to conformal time η, $X_0 = \varphi_0'^2/(2a^2)$ and we have set $G = 1$. To linear order the perturbations of the energy–momentum tensor (1.2) are:

$$\delta T_0^0 = \delta\varepsilon = \frac{\varepsilon + p}{c_s^2}\left(\left(\frac{\delta\varphi}{\varphi_0'}\right)' + \mathcal{H}\frac{\delta\varphi}{\varphi_0'} - \Phi\right) - 3\mathcal{H}(\varepsilon + p)\frac{\delta\varphi}{\varphi_0'}, \tag{2.4}$$

where

$$c_s^2 \equiv \frac{p_{,X}}{\varepsilon_{,X}} = \frac{\varepsilon + p}{2X\varepsilon_{,X}}, \tag{2.5}$$

and

$$\delta T_i^0 = (\varepsilon + p)u^0\delta u_i = (\varepsilon + p)g^{00}\frac{\varphi_0'}{\sqrt{2X_0}}\frac{\delta\varphi_{,i}}{\sqrt{2X_0}} = (\varepsilon + p)\left(\frac{\delta\varphi}{\varphi_0'}\right)_{,i}. \tag{2.6}$$

The $0-0$ and $0-i$ components of the Einstein equations then become

$$\Delta\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi a^2(\varepsilon + p)\left[\frac{1}{c_s^2}\left(\left(\frac{\delta\varphi}{\varphi_0'}\right)' + \mathcal{H}\frac{\delta\varphi}{\varphi_0'} - \Phi\right) - 3\mathcal{H}\frac{\delta\varphi}{\varphi_0'}\right], \tag{2.7}$$

$$(\Psi' + \mathcal{H}\Phi) = 4\pi a^2(\varepsilon + p)\left(\frac{\delta\varphi}{\varphi_0'}\right). \tag{2.8}$$

Since $\delta T_k^i = 0$ for $i \neq k$, we have $\Psi = \Phi$; the two equations above are sufficient to determine the gravitational potential and the perturbation of the scalar field. It is useful, however, to recast them in a slightly different, more convenient form. Using equation (2.8) to express Φ in terms of Ψ' and $\delta\varphi$ and substituting the result into (2.7), we obtain

$$\Delta\Psi = \frac{4\pi a^2(\varepsilon + p)}{c_s^2\mathcal{H}}\left(\mathcal{H}\frac{\overline{\delta\varphi}}{\varphi_0'} + \Psi\right)', \tag{2.9}$$

where the background equations (2.2) and (2.3) have also been used. Because $\Phi = \Psi$, equation (2.8) can be rewritten as

$$\left(a^2\frac{\Psi}{\mathcal{H}}\right)' = \frac{4\pi a^4(\varepsilon + p)}{\mathcal{H}^2}\left(\mathcal{H}\frac{\overline{\delta\varphi}}{\varphi_0'} + \Psi\right). \tag{2.10}$$

Finally, in terms of the new variables

$$u \equiv \frac{\Psi}{4\pi(\varepsilon + p)^{1/2}}, \qquad v \equiv \sqrt{\varepsilon_{,X}}a\left(\overline{\delta\varphi} + \frac{\varphi_0'}{\mathcal{H}}\Psi\right), \tag{2.11}$$

(2.9) and (2.10) take the form

$$c_s\Delta u = z\left(\frac{v}{z}\right)', \qquad c_s v = \theta\left(\frac{u}{\theta}\right)' \tag{2.12}$$

where

$$z \equiv \frac{a^2(\varepsilon + p)^{1/2}}{c_s \mathcal{H}}, \qquad \theta \equiv \frac{1}{c_s z} = \sqrt{\frac{8\pi}{3}}\frac{1}{a}\left(1 + \frac{p}{\varepsilon}\right)^{-1/2}. \tag{2.13}$$

2.2. Classical solutions

Substituting v from the second equation in (2.12) into the first gives a closed form second order differential equation for u:

$$u'' - c_s^2 \Delta u - \frac{\theta''}{\theta} u = 0. \tag{2.14}$$

Considering a *short-wavelength* plane wave perturbation with a wavenumber k $\left(c_s^2 k^2 \gg |\theta''/\theta|\right)$, we obtain in the WKB approximation

$$u \simeq \frac{C}{\sqrt{c_s}} \exp\left(\pm \mathrm{i}k \int c_s \,\mathrm{d}\eta\right), \tag{2.15}$$

where C is a constant of integration. The *long-wavelength* solution, valid for $c_s^2 k^2 \ll |\theta''/\theta|$, is

$$u = C_1\theta + C_2\theta \int_{\eta_0} \frac{\mathrm{d}\eta}{\theta^2} + O((k\eta)^2). \tag{2.16}$$

Given u, the gravitational potential can be inferred from the definition in (2.11):

$$\Phi = \Psi = 4\pi(\varepsilon + p)^{1/2} u \tag{2.17}$$

and a perturbation of the scalar field is calculated using (2.8):

$$\delta\varphi = \varphi_0' \frac{(a\Phi)'}{4\pi a^3(\varepsilon + p)} = \dot{\varphi}_0 \frac{(\dot{\Phi} + H\Phi)}{4\pi(\varepsilon + p)}. \tag{2.18}$$

Substituting (2.15) and (2.16) into equations (2.17) and (2.18), we have

$$\Phi \simeq 4\pi C \dot{\varphi}_0 \sqrt{\frac{p_{,X}}{c_s}} \exp\left(\pm \mathrm{i}k \int \frac{c_s}{a}\,\mathrm{d}t\right), \tag{2.19}$$

$$\delta\varphi \simeq C \sqrt{\frac{1}{c_s p_{,X}}} \left(\pm \mathrm{i}c_s \frac{k}{a} + H + \cdots\right) \exp\left(\pm \mathrm{i}k \int \frac{c_s}{a}\,\mathrm{d}t\right), \tag{2.20}$$

for a *short-wavelength* perturbation and

$$\Phi \simeq A \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\int a\,\mathrm{d}t\right) = A\left(1 - \frac{H}{a}\int a\,\mathrm{d}t\right), \tag{2.21}$$

$$\delta\varphi \simeq A\dot{\varphi}_0 \left(\frac{1}{a}\int a\,\mathrm{d}t\right), \tag{2.22}$$

where A is a constant of integration, in the *long-wavelength* limit respectively.

Let us first find how a perturbation behaves during inflation. It follows from (2.19) and (2.20) that in the short-wavelength regime both metric and scalar field perturbations oscillate. The amplitude of the metric perturbation is proportional to $\dot{\varphi}_0$ and it grows only slightly towards the end of inflation, while the amplitude of scalar field perturbation decays in inverse proportion to the scale factor. After a perturbation enters the long-wavelength regime it is described by formulae (2.21) and (2.22). These formulae are simplified during slow-roll. Integrating by parts and neglecting the decaying mode we find that to leading order

$$\Phi \simeq A(H^{-1})^{\bullet} = -A\frac{\dot{H}}{H^2}, \qquad \delta\varphi \simeq A\frac{\dot{\varphi}_0}{H}. \tag{2.23}$$

Result (2.23) is applicable only during inflation. After the slow-roll stage is over we must use formulae (2.21) and (2.22) directly. Inflation is usually followed by an oscillatory stage where the scale factor grows as some power of time, $a \propto t^p$, with p depending on the scalar field potential (for the quadratic potential $p = 2/3$ and for the quartic potential $p = 1/2$). Neglecting the decaying mode we obtain from (2.21) and (2.22)

$$\Phi \simeq \frac{A}{p+1}, \qquad \delta\varphi \simeq \frac{At\dot{\varphi}_0}{p+1}, \tag{2.24}$$

that is, the amplitude of the gravitational potential freezes out after inflation.

The scalar field finally converts its energy into ultra-relativistic matter corresponding to $p = 1/2$. This influences the perturbations only via the change of the effective equation of state and the resulting amplitude is

$$\Phi \simeq \tfrac{2}{3}A. \tag{2.25}$$

Using (2.23), we can express A in terms of $\delta\varphi$, $\dot{\varphi}_0$ and H at the moment of sound horizon crossing, when $c_s k \sim Ha$. For those perturbations which leave the horizon during inflation the final result is

$$\Phi \simeq \frac{2}{3}\left(H\frac{\delta\varphi}{\dot{\varphi}_0}\right)_{c_s k \sim Ha}. \tag{2.26}$$

Note that formula (2.26) can also be applied to calculate the perturbations in theories with a non-minimal kinetic term.

Starting with quantum fluctuations, the resulting amplitude of perturbations in the post-inflationary epoch can be fixed if we know $\delta\varphi$ at horizon crossing. The natural question arises: which variable plays the role of a canonical quantization variable? To derive the exact numerical coefficients we need a rigorous quantum theory.

2.3. Action and quantization

In order to construct a canonical quantization variable and properly normalize the amplitude of quantum fluctuations, we need the action for the cosmological perturbations. To obtain it one expands the action for the gravitational and scalar fields to second order in perturbations. After use of the constraints, the result is reduced to an expression containing only the physical degrees of freedom (see, for example, [5]). The steps are very cumbersome but fortunately they can be avoided because the action for the perturbations can be unambiguously inferred directly from the equations of motion (2.12) up to an overall time-independent factor. This factor can then be fixed by calculating the action in some simple limiting case. The first order action reproducing the equations of motion (2.12) is

$$S = \int \left[\left(\frac{v}{z}\right)' \hat{O}\left(\frac{u}{\theta}\right) - \frac{1}{2}c_s^2(\Delta u)\hat{O}u + \frac{1}{2}c_s^2 v\hat{O}v\right] \mathrm{d}\eta\, \mathrm{d}^3x, \tag{2.27}$$

where $\hat{O} \equiv \hat{O}(\Delta)$ is a time-independent operator to be determined. Using the first equation in (2.12) to express u in terms of $(v/z)'$, we obtain

$$S = \frac{1}{2}\int \left[z^2\left(\frac{v}{z}\right)'\frac{\hat{O}}{\Delta}\left(\frac{v}{z}\right)' + c_s^2 v\hat{O}v\right] \mathrm{d}\eta\, \mathrm{d}^3x. \tag{2.28}$$

Comparing action (2.28) in the limiting case $\dot{\varphi}_0/H \to 0$ to the action for a free scalar field in the de Sitter universe, we obtain $\hat{O} = \Delta$ and (2.28) becomes

$$S \equiv \int \mathcal{L}\, \mathrm{d}\eta\, \mathrm{d}^3x = \frac{1}{2}\int \left(v'^2 + c_s^2 v \Delta v + \frac{z''}{z} v^2 \right) \mathrm{d}\eta\, \mathrm{d}^3x, \tag{2.29}$$

after we drop the total derivative terms. Varying the action with respect to v we obtain

$$v'' - c_s^2 \Delta v - \frac{z''}{z} v = 0. \tag{2.30}$$

Note that this equation also follows from the second equation in (2.12) after substituting u in terms of v.

The quantization of cosmological perturbations with action (2.29) is thus formally equivalent to the quantization of a 'free scalar field' v with time-dependent 'mass' $m^2 = -z''/z$ in Minkowski space. The time dependence of the 'mass' is due to the interaction of the perturbations with the homogeneous expanding background. The energy of the perturbations is not conserved and they can be excited by borrowing energy from the Hubble expansion. The canonical quantization variable

$$v = \sqrt{\varepsilon_{,X}}\, a \left(\delta\varphi + \frac{\varphi_0'}{\mathcal{H}} \Psi \right) \tag{2.31}$$

is a gauge-invariant combination of the scalar field and metric perturbations. The operator $\hat{v}$ obeys equation (2.30) and its general solution can be written as

$$\hat{v}(\eta, \mathbf{x}) = \frac{1}{\sqrt{2}} \int \left[v_{\mathbf{k}}^*(\eta)\, \mathrm{e}^{\mathrm{i}\mathbf{kx}} \hat{a}_{\mathbf{k}}^- + v_{\mathbf{k}}(\eta)\, \mathrm{e}^{-\mathrm{i}\mathbf{kx}} \hat{a}_{\mathbf{k}}^+ \right] \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}, \tag{2.32}$$

where $\hat{a}_{\mathbf{k}}^+$ and $\hat{a}_{\mathbf{k}}^-$ are the creation and annihilation operators which satisfy the standard commutation relations. The temporal mode functions $v_{\mathbf{k}}(\eta)$ obey the equation

$$v_{\mathbf{k}}'' + \omega_k^2(\eta) v_{\mathbf{k}} = 0, \qquad \omega_k^2(\eta) \equiv c_s^2 k^2 - z''/z. \tag{2.33}$$

with initial conditions

$$v_{\mathbf{k}}(\eta_i) = \frac{1}{\sqrt{\omega_k(\eta_i)}}, \qquad v_{\mathbf{k}}'(\eta_i) = \mathrm{i}\sqrt{\omega_k(\eta_i)}. \tag{2.34}$$

Note that the above conditions make sense only for modes with $c_s^2 k^2 > (z''/z)_i$ for which $\omega_k^2(\eta_i) > 0$.

The next step in quantization is to define the 'vacuum' state $|0\rangle$ as the state annihilated by operators $\hat{a}_{\mathbf{k}}^-$:

$$\hat{a}_{\mathbf{k}}^- |0\rangle = 0. \tag{2.35}$$

We further assume that a complete set of independent states in the corresponding Hilbert space can be obtained by acting with the products of creation operators on the vacuum state $|0\rangle$. If the ω_k do not depend on time, then the vector $|0\rangle$ corresponds to the familiar Minkowski vacuum. Assuming c_s changes adiabatically, we find that modes with $c_s^2 k^2 \gg (z''/z)$ remain unexcited and minimal fluctuations are well-defined. On the other hand, for modes with $c_s^2 k^2 < (z''/z)_i$ we have $\omega_k^2(\eta_i) < 0$, and the initial minimal fluctuations on corresponding scales cannot be unambiguously determined. These scales exceed the Hubble scale at the beginning of inflation and are subsequently stretched to huge unobservable scales; therefore the question of initial fluctuations here is fortunately moot. The inhomogeneities responsible for the observable structure originate from quantum fluctuations on scales where the minimal fluctuations are unambiguously defined.

2.4. Spectrum

Our final task is to calculate the correlation function, or equivalently, the power spectrum of the gravitational potential. Taking into account (2.11), we have the following expansion for the operator $\hat{\Phi}$:

$$\hat{\Phi}(\eta,\mathbf{x}) = \frac{4\pi(\varepsilon+p)^{1/2}}{\sqrt{2}}\int\left[u_{\mathbf{k}}^{*}(\eta)\,\mathrm{e}^{\mathrm{i}\mathbf{kx}}\hat{a}_{\mathbf{k}}^{-}+u_{\mathbf{k}}(\eta)\,\mathrm{e}^{-\mathrm{i}\mathbf{kx}}\hat{a}_{\mathbf{k}}^{+}\right]\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}, \qquad (2.36)$$

where the mode functions $u_{\mathbf{k}}(\eta)$ obey (2.14) and are related to the mode functions $v_{\mathbf{k}}(\eta)$ via (2.12). For the initial vacuum state $|0\rangle$ the correlation function at $\eta > \eta_i$ is

$$\langle 0|\hat{\Phi}(\eta,\mathbf{x})\hat{\Phi}(\eta,\mathbf{y})|0\rangle = \int 4(\varepsilon+p)|u_k|^2k^3\frac{\sin kr}{kr}\frac{\mathrm{d}k}{k}, \qquad (2.37)$$

where $r \equiv |\mathbf{x}-\mathbf{y}|$. According to the definition of the power spectrum, we have

$$\delta_\Phi^2(k,\eta) = 4(\varepsilon+p)|u_{\mathbf{k}}(\eta)|^2k^3. \qquad (2.38)$$

Given $v_{\mathbf{k}}(\eta_i)$ and $v_{\mathbf{k}}'(\eta_i)$, the initial conditions for $u_{\mathbf{k}}$ can be inferred from equations (2.12). Let us consider a short-wavelength perturbation with $c_s^2k^2 \gg (z''/z)_i$ for which $\omega_k(\eta_i) \simeq c_s k$. In this case the initial conditions (2.34) can be rewritten in terms of $u_{\mathbf{k}}$ as

$$u_{\mathbf{k}}(\eta_i) \simeq -\frac{\mathrm{i}}{\sqrt{c_s}k^{3/2}}, \qquad u_{\mathbf{k}}'(\eta_i) \simeq \frac{\sqrt{c_s}}{k^{1/2}}, \qquad (2.39)$$

where we have neglected higher order terms, which are suppressed by powers of $(c_s k\eta_i)^{-1} \ll 1$. The corresponding short-wavelength WKB solution, valid for $c_s^2k^2 \gg |\theta''/\theta|$, is

$$u_{\mathbf{k}}(\eta) \simeq -\frac{\mathrm{i}}{\sqrt{c_s}k^{3/2}}\exp\left(\mathrm{i}k\int_{\eta_i}^{\eta}c_s\,\mathrm{d}\tilde{\eta}\right). \qquad (2.40)$$

During inflation the ratio $|\theta''/\theta|$ can be estimated roughly as $\eta^{-2}|\dot{H}/H^2|$. Because $|\dot{H}/H^2| \ll 1$, formula (2.40) is still applicable within the short time interval

$$\frac{1}{c_s k} > |\eta| > \frac{1}{k}|\dot{H}/H^2|^{1/2} \qquad (2.41)$$

after the sound horizon crossing. At this time the argument in the exponent is almost constant and $u_{\mathbf{k}}$ freezes out. After a perturbation enters the long-wavelength regime the time evolution of the gravitational potential is described by (2.21), and hence

$$u_{\mathbf{k}}(\eta) \equiv \frac{\Phi}{4\pi(\varepsilon+p)^{1/2}} = \frac{A_{\mathbf{k}}}{4\pi(\varepsilon+p)^{1/2}}\left(1-\frac{H}{a}\int a\,\mathrm{d}t\right). \qquad (2.42)$$

During inflation this expression simplifies to

$$u_{\mathbf{k}}(\eta) \simeq -\frac{A_{\mathbf{k}}}{4\pi(\varepsilon+p)^{1/2}}\left(\frac{\dot{H}}{H^2}\right) = A_{\mathbf{k}}\frac{(\varepsilon+p)^{1/2}}{H^2}. \qquad (2.43)$$

Taking into account that within the time interval (2.41) the ratio

$$\frac{(\varepsilon+p)^{1/2}}{H^2}$$

is almost constant and comparing solutions (2.40) and (2.43), we obtain

$$A_{\mathbf{k}} \simeq -\frac{\mathrm{i}}{k^{3/2}}\left(\frac{H^2}{\sqrt{c_s}(\varepsilon+p)^{1/2}}\right)_{c_sk\simeq Ha}. \qquad (2.44)$$

Substituting (2.40) into (2.38) gives the scale-independent power spectrum

$$\delta_\Phi^2(k,t) \simeq \frac{4(\varepsilon+p)}{c_s} \qquad (2.45)$$

for short-wavelength perturbations with $k > Ha(t)/c_s$. Using (2.42) with $A_{\mathbf{k}}$ as given in (2.44), we obtain

$$\delta_\Phi^2(k,t) \simeq \frac{16}{9}\left(\frac{\varepsilon}{c_s(1+p/\varepsilon)}\right)_{c_s k \simeq Ha}\left(1-\frac{H}{a}\int a\,\mathrm{d}t\right)^2 \tag{2.46}$$

for long-wavelength perturbations with $Ha(t)/c_s > k > Ha_i/c_s$, where $a_i \equiv a(t_i)$.

It follows from (2.46) that in the post-inflationary, radiation-dominated epoch the resulting power spectrum is

$$\delta_\Phi^2 \simeq \frac{64}{81}\left(\frac{\varepsilon}{c_s(1+p/\varepsilon)}\right)_{c_s k \simeq Ha}. \tag{2.47}$$

This formula is applicable only on scales corresponding to $\left(c_s^{-1}Ha\right)_f > k > \left(c_s^{-1}Ha\right)_i$. This range surely encompasses the observable universe. The supercurvature perturbations are frozen during the radiation-dominated stage and they survive unchanged until recombination. Only for those scales which reenter the horizon does the evolution proceed in a nontrivial way.

In particular case of an inflationary model with potential $V = (\lambda/n)\varphi^n$ we obtain

$$\delta_\Phi^2 \simeq \frac{128}{27}\frac{n^{\frac{n}{2}-2}}{(4\pi)^{n/2}}\lambda[\ln(\lambda_{ph}/\lambda_\gamma)]^{\frac{n+2}{2}}, \tag{2.48}$$

where λ_γ is the typical wavelength of the background radiation.

2.5. Spectral tilt

It follows from (2.47) that the amplitude of the metric perturbation on a given comoving scale is determined by the energy density and by deviation of the equation of state from the vacuum equation of state at the time of horizon crossing. On galactic scales, δ_Φ^2 is of order 10^{-10} and $(1+p/\varepsilon)$ can be estimated as $\sim 10^{-2}$; therefore we conclude that $\varepsilon \sim 10^{-12}$ of the Planckian density at this time. This is a rather robust and generic estimate for inflation during the last seventy e-folds. Only if $c_s \ll 1$, for instance in k-inflation, can we avoid this conclusion.

Since inflation must have a graceful exit, the energy density and the equation of state slowly change during inflation. As a consequence the amplitude of the perturbations generated depends slightly on lengthscale. The energy density always decreases and it is natural to expect that the deviation of the equation of state from that for the vacuum should increase towards the end of inflation. It follows then from (2.47) that the amplitude of those perturbations which crossed the horizon earlier must be larger than the amplitude of perturbations which crossed later. Within a narrow range of scales, one can always approximate the spectrum by the power-law, $\delta_\Phi^2(k) \propto k^{n_S-1}$, and thus characterize it by the spectral index n_S. A flat spectrum corresponds to $n_S = 1$.

The expression for the spectral index follows from (2.47):

$$n_S - 1 \equiv \frac{\mathrm{d}\ln\delta_\Phi^2}{\mathrm{d}\ln k} \simeq -3\left(1+\frac{p}{\varepsilon}\right) - \frac{1}{H}\left(\ln\left(1+\frac{p}{\varepsilon}\right)\right)^{\bullet} - \frac{(\ln c_s)^{\bullet}}{H}, \tag{2.49}$$

where the quantities on the right-hand side must be calculated at the time of horizon crossing. In deriving this formula we have taken into account that $\mathrm{d}\ln k \simeq \mathrm{d}\ln a_k$. This relation follows from the condition determining horizon crossing, $c_s k \simeq Ha_k$, if we neglect the change in c_s and H. All terms on the right-hand side of (2.49) are negative for a generic inflationary scenario. Therefore, inflation does not predict a flat spectrum, as is quite often mistakenly stated. Instead, it predicts a red-tilted spectrum: $n_S < 1$ so that the amplitude grows slightly towards the larger scales. The physical reason for this tilt is the necessity of a smooth graceful exit. To obtain an estimate for the tilt we note that the galactic scales cross the horizon around

50 to 60 e-folds before the end of inflation. At this time $(1 + p/\varepsilon)$ is larger than 10^{-2}. The second term in (2.49) is about the same order of magnitude and the spectral index can thus be estimated as $n_S \simeq 0.96$. The concrete value of n_S depends on a particular inflationary scenario. Even without knowing this scenario, however, one could expect that $n_S \leqslant 0.97$. By inspection of the variety of scenarios, one infers that it is rather difficult to get a very large deviation from the flat spectrum and that it is likely $n_S > 0.92$.

Considering chaotic inflation in a model with potential V we find

$$n_S - 1 \simeq -\frac{3}{8\pi}\left(\frac{V_{,\varphi}}{V}\right)^2 + \frac{1}{4\pi}\frac{V_{,\varphi\varphi}}{V}. \tag{2.50}$$

and in particular for the power-law potential, $V \propto \varphi^n$,

$$n_S - 1 \simeq -\frac{n(n+2)}{8\pi\varphi^2_{k\simeq Ha}} \simeq -\frac{n+2}{2N}, \tag{2.51}$$

where N is the number of e-folds before the end of inflation when the corresponding perturbation crosses the horizon. In the case of a massive scalar field, $n = 2$, and $n_S \simeq 0.96$ on galactic scales for which $N \simeq 50$. For the quartic potential $n = 4$ and $n_S \simeq 0.94$.

3. Gravitational waves

In a similar manner to scalar perturbations, long-wavelength gravitational waves are also generated in inflation [16]. In this case the calculations are not very different from those made for the scalar perturbations and are in fact much easy. First we need the action for the gravitational waves, described by transverse, traceless part of metric perturbations h_{ik}, is (see, for example, [5]):

$$S = \frac{1}{64\pi}\int a^2\left(h^{i\prime}_{j} h^{j\prime}_{i} - h^{i}_{j,l} h^{j,l}_{i}\right) \mathrm{d}\eta\, \mathrm{d}^3x, \tag{3.1}$$

where the spatial indices are raised and lowered with the help of the unit tensor δ_{ik}. Substituting

$$h^i_j(\mathbf{x}, \eta) = \int h_{\mathbf{k}}(\eta) e^i_j(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{kx}} \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}, \tag{3.2}$$

where $e^i_j(\mathbf{k})$ is the polarization tensor, into (3.1), we obtain

$$S = \frac{1}{64\pi}\int a^2 e^i_j e^j_i \left(h'_{\mathbf{k}} h'_{-\mathbf{k}} - k^2 h_{\mathbf{k}} h_{-\mathbf{k}}\right) \mathrm{d}\eta\, \mathrm{d}^3k. \tag{3.3}$$

Rewritten in terms of the new variable

$$v_{\mathbf{k}} = \sqrt{\frac{e^i_j e^j_i}{32\pi}}\, a h_{\mathbf{k}}, \tag{3.4}$$

the action becomes

$$S = \frac{1}{2}\int \left(v'_{\mathbf{k}} v'_{-\mathbf{k}} - \left(k^2 - \frac{a''}{a}\right) v_{\mathbf{k}} v_{-\mathbf{k}}\right) \mathrm{d}\eta\, \mathrm{d}^3k. \tag{3.5}$$

It describes a real scalar field in terms of its Fourier components. The resulting equations of motion are

$$v''_{\mathbf{k}} + \omega^2_k(\eta) v_{\mathbf{k}} = 0, \qquad \omega^2_k(\eta) \equiv k^2 - a''/a. \tag{3.6}$$

There is no need to repeat the quantization procedure for this case. Taking into account (3.4) and (3.2), we immediately find the correlation function

$$\langle 0|h^i_j(\eta, \mathbf{x}) h^j_i(\eta, \mathbf{y})|0\rangle = \frac{8}{\pi a^2}\int |v_{\mathbf{k}}|^2 k^3 \frac{\sin kr}{kr}\frac{\mathrm{d}k}{k}, \tag{3.7}$$

where $v_{\mathbf{k}}$ is the solution of equation (3.6) with initial conditions

$$v_{\mathbf{k}}(\eta_i) = \frac{1}{\sqrt{\omega_k}}, \qquad v'_{\mathbf{k}}(\eta_i) = \mathrm{i}\sqrt{\omega_k}. \tag{3.8}$$

These initial conditions make sense only if $\omega_k > 0$, that is, for gravitational waves with $k^2 > (a''/a)_{\eta_i}$. The power spectrum, characterizing the strength of a gravitational wave with comoving wavenumber k, is correspondingly

$$\delta_h^2(k, \eta) = \frac{8|v_{\mathbf{k}}|^2 k^3}{\pi a^2}. \tag{3.9}$$

3.1. Inflation

In contrast to scalar perturbations, the deviation of the equation of state from the vacuum equation of state is not so crucial to the evolution of gravitational waves. Therefore, we first consider a pure de Sitter universe where $a = -(H_\Lambda \eta)^{-1}$. In this case equation (3.6) simplifies to

$$v''_{\mathbf{k}} + \left(k^2 - \frac{2}{\eta^2}\right) v_{\mathbf{k}} = 0. \tag{3.10}$$

Let us consider gravitational waves with $k|\eta_i| \gg 1$ for which $\omega_k \simeq k$. Taking into account the initial conditions in (3.8), the solution of this equation becomes

$$v_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{k}} \left(1 + \frac{\mathrm{i}}{k\eta}\right) \exp(\mathrm{i}k(\eta - \eta_i)). \tag{3.11}$$

and hence

$$\delta_h^2 = \frac{8H_\Lambda^2}{\pi}[1 + (k\eta)^2] = \frac{8H_\Lambda^2}{\pi}\left[1 + \left(\frac{k_{ph}}{H_\Lambda}\right)^2\right], \tag{3.12}$$

where $k_{ph} \equiv k/a$ is the physical wavelength. This formula is applicable only for $k_{ph} \gg H_\Lambda(\eta/\eta_i)$. The long-wavelength gravitational waves with $H_\Lambda^{-1}(\eta_i/\eta) > \lambda_{ph} > H_\Lambda^{-1}$ have a flat spectrum with amplitude proportional to H_Λ.

The above consideration refers to a pure de Sitter universe where H_Λ is exactly constant. In realistic inflationary models the Hubble constant slowly changes with time. Recalling that the non-decaying mode of a gravitational wave is frozen on supercurvature scales, we obtain

$$\delta_h^2 \simeq \frac{8H_{k\simeq Ha}^2}{\pi} = \frac{64}{3}\varepsilon_{k\simeq Ha}. \tag{3.13}$$

The tensor spectral index is then equal to

$$n_T \equiv \frac{\mathrm{d}\ln\delta_h^2}{\mathrm{d}\ln k} \simeq -3\left(1 + \frac{p}{\varepsilon}\right)_{k\simeq Ha}, \tag{3.14}$$

and hence the spectrum of the gravitational waves is also slightly tilted to the red. (Note that the tensor and scalar spectral indices are defined differently—see equation (2.49).) The ratio of tensor to scalar power spectrum amplitudes on supercurvature scales during the post-inflationary, radiation-dominated epoch is

$$\frac{\delta_h^2}{\delta_\Phi^2} \simeq 27\left[c_s\left(1 + \frac{p}{\varepsilon}\right)\right]_{k\simeq Ha}. \tag{3.15}$$

For a canonical scalar field ($c_s = 1$), this ratio is generically about 20 to 30%. However, in k-inflation, where $c_s \ll 1$, it can be strongly suppressed. Thus, at least in principle, k-inflation is phenomenologically distinguishable from inflation based on a scalar field potential.

4. Inflation as a theory with predictive power

Assuming a stage of cosmic acceleration—inflation—we are able to make robust predictions even in the absence of the actual inflationary scenario. The most important among them are:

(*i*) *the flatness of the universe*;
(*ii*) *Gaussian scalar metric perturbations with a slightly red-tilted spectrum*;
(*iii*) *long-wavelength gravitational waves*.

The condition of flatness is not as 'natural' as it might appear at first glance. We recall that $\Omega_0 = 1$ was strongly disfavoured by observations not so long ago. If gravity were always an attractive force, it is absolutely unclear why the current value of Ω_0 could not be, for instance, 0.01 or 0.2. Only inflation gives a natural justification for $\Omega_0 = 1$. The deuterium abundance clearly indicates that baryons cannot contribute more than a few per cent to the critical energy density. Therefore, inflation also predicts the existence of a dark component. It can be dark matter, dark energy or the combination of the two. In the absence of the actual inflationary scenario, we cannot make any prediction about the composition of the dark component. In spite of the tremendous progress made recently, we are still far from understanding the true nature of dark matter and dark energy. The current data on CMB fluctuations favour the critical density and, combined with the results from high-redshift supernovae, make it almost impossible to doubt the existence of dark matter and dark energy.

The predicted spectrum for the scalar perturbations is also in good agreement with the current data. However, the accuracy of the observations is not yet sufficient to determine a small spectral tilt. The deviation of the spectrum from flat is an inevitable consequence of simple inflation and therefore it is extremely important to detect it. The amplitude of the power spectrum is a free parameter of the theory.

The production of a significant amount of long-wavelength gravitational waves is another generic prediction of a broad class of simple inflationary scenarios. While their detection would strongly support inflation, the absence of gravitational waves would not allow us to exclude simple inflation since their production can be avoided in *k*-inflation.

Since we do not know which concrete scenario was realized in nature, the question of the robustness of the predictions of inflation is of particular importance. Simple inflation does not leave much room for ambiguities. However, it is clear that by introducing extra parameters and by fine-tuning, one can spoil practically any prediction of the theory. For example, with the help of a second inflationary stage, we could avoid the flatness constraint [17]. Similarly, by involving two or more scalar fields, one can obtain practically any spectrum of cosmological perturbations and induce nongaussianity [18–20]. In these cases inflation loses its predictive power and becomes not so attractive. Having said this, the 'price to performance' ratio for these models seems too high to consider them realistic descriptions of nature. Therefore, only observations confirming the robust predictions of inflation can assure us that we are on the right track in understanding our universe. Otherwise they would simply open the door for speculations.

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